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Research article

Homotopy Perturbation Method and its Application in Solving Nonlinear Problems

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Abstract

The Homotopy Perturbation Method (HPM) is a well-known mathematical approach utilized to solve many different type of differential equation. This method involves constructing a homotopy, a continuous transformation, between a known initial problem and the desired nonlinear problem. Perturbation terms are then introduced to iteratively approximate the solution. In this paper, we explore the fundamentals of the HPM and its application in tackling nonlinear problems across various fields, including physics, engineering, and mathematical modelling. Through illustrative examples and case studies, we demonstrate the effectiveness and versatility of the HPM in providing accurate solutions to nonlinear problems that are otherwise challenging to solve using traditional methods. Additionally, we discuss the advantages, limitations, and future directions of the HPM in non-linear space.

1. Numerical Analysis:

Numerical Analysis is a field of mathematics that focuses on solving mathematical problems through computational methods. Its primary objective is to address various mathematical challenges by employing arithmetic operations and algorithms. One of the fundamental tasks of numerical analysis is finding roots of algebraic equations, which involves determining the values of variables that satisfy the equation. Additionally, it facilitates the calculation of derivatives of functions, allowing for the determination of the rate of change of a function at a given point. Moreover, numerical analysis is instrumental in the integration of functions, enabling the computation of definite integrals and the calculation of areas under curves. It is also extensively used in solving both ordinary and partial differential equations, which are ubiquitous in various scientific and engineering fields. By discretizing these equations and applying numerical methods, approximate solutions can be obtained, aiding in the understanding and prediction of complex physical phenomena. We have the significant reward of numerical analysis is its ability to offer approximate solutions to problems that may not have exact analytical solutions. This is particularly valuable in scenarios where analytical methods are infeasible or impractical. Furthermore, numerical analysis allows for the optimization of computational resources by adjusting parameters such as step size to minimize errors in the results. In essence, numerical analysis plays a vital role in modern mathematics, science, and engineering by providing computational tools and techniques to tackle a wide range of mathematical problems efficiently and accurately (see [1]-[9]).

The structure of this article is organized into seven distinct sections. Section 1 provides a definition of Numerical Analysis, outlining its fundamental concepts and methodologies. Moving forward, Section 2 delves into the Applications of Numerical Analysis, exploring its practical significance across various fields. In Section 3, the focus shifts to the discussion of the generalized Burgers equation, elucidating its mathematical formulation and significance in fluid dynamics and nonlinear physics. Section 4 introduces the Homotopy Perturbation Method (HPM), presenting its principles and techniques for solving nonlinear problems. This section also outlines the procedure for applying HPM, detailing its step-by-step approach. Section 5 demonstrates the application of HPM in solving the generalized Burgers equation through a basic example, showcasing the method's effectiveness in obtaining approximate solutions. In Section 6, specific cases derived from the generalized Burgers equation are examined, providing insights into the equation's behavior under various conditions. Lastly, Section 7 serves as the conclusion of the paper, summarizing key findings and insights gleaned from the discussion. Additionally, it discusses the implications of the research and avenues for future exploration in the field.

2. **Applications of Numerical Analysis:**

Numerical analysis holds a critical position within mathematics, exerting significant influence across diverse scientific and technological realms, particularly in addressing differential equations. A plethora of methods exist in literature for tackling these equations, encompassing techniques such as the Method of Lines, Finite Difference Method, Gradient Discretization

Method, Finite Element Method, Finite Volume Method, Euler's Method, Spectral Method, Improved Euler Method, Runge-Kutta Method, Parallel-in-Time Methods, Finite Differences, Galerkin Methods, Adams–Bashforth Methods, Homotopy Analysis Method, Adomian Decomposition Method, and more. This paper specifically concentrates on the Homotopy Perturbation Method and its utilization, with a primary focus on solving nonlinear equations. These nonlinear differential equations are fundamental in elucidating a myriad of physical phenomena.

3. **Generalized Burgers Equation:**

Calculus has experienced significant growth in acceptance and status over the past few years across several fields of engineering and science. Its increasing the number of requests highlights calculus's efficacy in providing enhanced mathematical mockups for understanding real-world substances and developments. Mathematical modeling, facilitated by calculus, plays a crucial role in delineating physical and natural phenomena, thereby aiding in problem analysis. The literature on calculus continues to expand rapidly due to ongoing global research efforts. Its impact spans numerous areas, including mechanics, biology, electricity, fluid dynamics, control theory, heat conduction, sports, viscoelasticity, image processing, and astrophysics (Ros, 2008; Husain, 2019). Consequently, calculus of integer order permeates every dimension of research and technology.

Ordinary differential equations (ODEs) include real integer order differential operator and often arise when solving various physical marvels. However, in many cases, ODEs have exact answers. Consequently, numerous methods are established to find the approximate solutions accurately and efficiently. Among these methods, the Homotopy Perturbation Method (HPM) stands out for its simplicity and superior convergence rate. Introduced by He (He, 1999), HPM has become widely utilized for solving nonlinear problems due to its effectiveness. One of its key advantages lies in the rapid convergence of the series solution, often requiring only a few iterations to achieve highly accurate results.

In this study, we explore the Burgers equation and its outcome using the homotopy perturbation method. That equation, initially presented by Bateman (Bateman, 1915) and further studied in 1948 by Burgers, represents a fundamental equation in fluid dynamics and nonlinear partial differential equations. For the detail about the Burgers equation (see [10]-[22]) can be expressed as:

$$\frac{\partial v}{\partial \beta} + av\frac{\partial v}{\partial \rho} = c\frac{\partial^2 v}{\partial \rho^2}. \qquad (1)$$

where, a is any arbitrary constant. The precise solution to the aforementioned equation can be derived using various methods such as the Cole-Hopf transform method, the tanh-coth method, and the Exp-function method etc.

In this paper, our focus will be on examining a generalized Burgers equation formulated in terms of the generalized fractional derivative.

$$\frac{\partial v}{\partial \rho} + av^m \frac{\partial v}{\partial \beta} = c\frac{\partial^2 v}{\partial \beta^2} \tag{2}$$

with

$$v(\beta, 0) = g(\beta),\ \beta \in \Omega\,. \tag{3}$$

Where $m \in N$.

We denote equation (2) as the generalized Burgers equation. To obtain an approximate symmetric solution under the given condition (3), we utilize the widely recognized homotopy perturbation method (HPM) to solve equation (2).

4. **Homotopy Perturbation Method (HPM):**

The HPM is a powerful mathematical procedure for finding the solution of nonlinear differential equations. It is constructed on the idea of homotopy, which is a continuous transformation from one equation to alternative. Here's a detailed explanation and solution procedure for applying homotopy perturbation method (HPM):

(a) **Formulate the Differential Equation (DE):**

Start with the nonlinear differential equation you want to solve. It could be an ordinary differential equation (ODE) or a partial differential equation (PDE). Let's denote this equation as

$F\left(x, u, u', u'', \ldots, u(n)\right) = 0,$

where u is the dependent variable, and u′, u″, etc., denote its derivatives with respect to the independent variable x.

Construct the Homotopy Operator: Introduce an auxiliary parameter τ and construct a homotopy operator L such that:

$$L\left(u(x); \tau\right) = \tau G\left(u(x)\right) + (1-\tau) F\left(x, u(x)\right) = 0 \tag{4}$$

Here, $F\left(x,\ u(x)\right)$ is the original differential equation, and G(u(x)) is a known function that makes the equation linearly solvable.

Assume the Perturbation Series Solution: Assume a solution u(x) in the form of a perturbation series:

$$u(x)=\sum_{n=0}^{\infty}u_n(x)\tau^n \tag{5}$$

where $u_n(x)$ are unknown functions to be determined.

(b) Substitute the Series Solution into the Homotopy Operator:

Substitute the perturbation series solution into the homotopy operator equation $L\left(u(x);\ \tau\right)=0$ and expand it in powers of τ. Equate coefficients of like powers of τ to obtain a sequence of equations for each order n.

(c) Solve the Perturbation Equations:

Solve the sequence of equations obtained in the previous step iteratively to find successive approximations $u_n(x)$. This can be done using various techniques such as power series methods, iteration methods, or numerical methods.

(d) **Convergence Analysis:**

Examine the convergence behavior of the series solution. Determine the range of validity and accuracy of the solution by investigating the convergence of the series. It's important to check the convergence criterion to ensure the solution is reliable.

(e) Check for Special Cases and Simplifications:

Depending on the problem's complexity, you may need to consider special cases or make simplifications to facilitate the solution process. This could involve choosing appropriate boundary or initial conditions or making assumptions to simplify the problem.

(f) **Verify the Solution:**

Once the solution is obtained, verify it by substituting it back into the original differential equation to ensure that it satisfies the equation within the desired accuracy.

By following these steps, you can apply the Homotopy Perturbation Method to solve a wide range of nonlinear differential equations efficiently and accurately. It's a flexible and robust method, particularly suitable for problems where traditional analytical or numerical techniques may fail. However, careful consideration of convergence properties and suitable initial approximations is essential for obtaining accurate solutions.

(g) Generalized Burgers equation and its Solution:

When confronting linear or non-linear problems, researchers have contributed a plethora of methods over time to address these challenges. Among the array of techniques available, the Homotopy Perturbation Method (HPM) and other methods (see [23]-[27]) emerges as a particularly robust tool for seeking solutions. In this section, we delve into the intricacies of the

HPM. Our intention is to apply this method to the specified problems outlined in equations (2) and (3). Consequently, we proceed to visualize the function v in the following manner

$v(\rho,\beta;\tau):\Xi\times[0,T]\times[0,1]\to R$

such as

$$H\left(v(\rho,\beta;\tau),\tau\right)=(1-\tau)\left[D_{\beta}v(\rho,\beta;\tau)-D_{\beta}u_{0}(\rho,\beta)\right] \\ +\tau\left[D_{\beta}v(\rho,\beta;\tau)+av^{m}(\rho,\beta;\tau)\frac{\partial v(\rho,\beta;\tau)}{\partial\rho}-c\frac{\partial^{2}v(\rho,\beta;\tau)}{\partial\rho^{2}}\right]=0, \tag{6}$$

Here, τ represents an embedding parameter, and $v_0(\rho,\beta)$ epitomizes an primary approximation. Now, by employing the preceding calculation, equation (4), we obtain:

$$D_{\beta}v(\rho,\beta;\tau)=D_{\beta}u_{0}(\rho,\beta) \\ -\tau\left[D_{\beta}u_{0}(\rho,\beta)+av^{m}(\rho,\beta;\tau)\frac{\partial v(\rho,\beta;\tau)}{\partial\rho}-c\frac{\partial^{2}v(\rho,\beta;\tau)}{\partial\rho^{2}}\right]. \tag{7}$$

Now, putting $v(\rho,\beta;\tau)=\sum_{l=0}^{\infty}\tau^{l}v_{l}(\rho,\beta)$ in equation (5), we get

$$D_{\beta}\sum_{k=0}^{\infty}\tau^{k}v_{k}(\rho,\beta)=D_{\beta}u_{0}(\rho,\beta)-\tau\left[D_{\beta}^{\mu}u_{0}(\rho,\beta)+a\left(\sum_{k=0}^{\infty}\tau^{k}v_{k}(\rho,\beta)\right)^{m}\right. \\ \left.\cdot\frac{\partial}{\partial\rho}\left(\sum_{k=0}^{\infty}\tau^{k}v_{k}(\rho,\beta)\right)-c\frac{\partial^{2}}{\partial\rho^{2}}\left(\sum_{k=0}^{\infty}\tau^{k}v_{k}(\rho,\beta)\right)\right]. \tag{8}$$

By attempting to equivalence the coefficients of the corresponding powers of τ in equation (6), we obtain:

$$\tau^{0}:D_{\beta}v_{0}(\rho,\beta)=D_{\beta}u_{0}(\rho,\beta),$$

and

$$\tau^{1}:D_{\beta}v_{1}(\rho,\beta)=-\left(D_{\beta}u_{0}(\rho,\beta)+av_{0}^{m}(\rho,\beta)\frac{\partial v_{0}(\rho,\beta)}{\partial\rho}\right)+c\frac{\partial^{2}v_{0}(\rho,\beta)}{\partial\rho^{2}},$$

also

$$\tau^{2}:D_{\beta}v_{2}(\rho,\beta)=-a\left[v_{0}^{m}(\rho,\beta)\frac{\partial v_{1}(\rho,\beta)}{\partial\rho}+mv_{0}^{m-1}(\rho,\beta)v_{1}(\rho,\beta)\frac{\partial v_{0}(\rho,\beta)}{\partial\rho}\right] \\ +c\frac{\partial^{2}v_{1}(\rho,\beta)}{\partial\mu^{2}},$$

On taking the integral, we get

$$v_{0}(\rho,\beta)=I_{\beta}\left(D_{\beta}u_{0}(\rho,\beta)\right),$$

or

$$v_0(\rho,\beta)=u_0(\rho,\beta),$$

or

$$v_0(\rho,0)=g(\rho).$$

and

$$v_1(\rho,\beta)=-I_\beta\left[\left(D_\beta u_0(\rho,\beta)+av_0^m(\rho,\beta)\frac{\partial v_0(\rho,\beta)}{\partial\rho}\right)-\right.$$
$$\left. c\frac{\partial^2 v_0(\rho,\beta)}{\partial\rho^2}\right],\ v_1(\rho,0)=0.$$

also

$$v_2\left(\rho,\beta\right)=-I_\beta\left(a\left[v_0^m(\rho,\beta)\frac{\partial v_1(\rho,\beta)}{\partial\rho}+\right.\right.$$
$$\left.\left. mv_0^{m-1}(\rho,\beta)v_1(\rho,\beta)\frac{\partial v_0(\rho,\beta)}{\partial\rho}\right]+c\frac{\partial^2 v_1(\rho,\beta)}{\partial\rho^2}\right),$$

After obtaining these values, we can determine the result v by substituting them into the following power series.

$$v=v_0+\tau\ v_1+\tau^2 v_2+\tau^3 v_3+\ldots \tag{9}$$

When obtaining the outcomes, if we consider the limit $\tau\to 1\ in\ v(\rho,\beta;\tau),$ we arrive at $v(\rho,\beta)$ as described in equations (9).

(h) Particular Case:

In this segment, we discuss the few particular cases of the generalized Burgers equation. We have considered the particular values of constant:

Example (1):

The objective of this section is to apply HPM, led earlier, to crack a precise example of the generalized Burgers equation. Through the utilization of HPM, I aim to derive an approximate solution for this particular example of the generalized Burgers equation, as defined by (2) and (3).

We have

$$D_\beta v(\rho,\beta)=-av(\rho,\beta)\frac{\partial v(\rho,\beta)}{\partial\rho}+c\frac{\partial^2 v(\rho,\beta)}{\partial\rho^2}, \tag{10}$$

with the initial condition

$$v_0=u_0=\rho^2. \tag{11}$$

With the help of homotopy technique, we obtain

$$D_\beta v(\rho,\beta;\tau)=D_\beta u_0(\rho,\beta)$$
$$-\tau\left(D_\beta u_0(\rho,\beta)+av(\rho,\beta;\tau)v(\rho,\beta;\tau)\frac{\partial v}{\partial\rho}-c\frac{\partial^2 v(\rho,\beta;\tau)}{\partial\rho^2}\right). \tag{12}$$

Let us consider

$$v(\rho,\beta;\tau)=\sum_{l=0}^{\infty}\tau^l v_l(\rho,\beta). \tag{13}$$

Now, using the same approach defined in above part, we obtain

$$\tau^0: D_\beta v_0(\rho,\beta)=D_\beta u_0(\rho,\beta),$$

and coefficient of τ

$$\tau^1: D_\beta v_1(\rho,\beta)=-\left(D_\beta u_0(\rho,\beta)+av_0(\rho,\beta)\frac{\partial v_0(\rho,\beta)}{\partial\rho}\right)+c\frac{\partial^2 v_0(\rho,\beta)}{\partial\rho^2},$$

and coefficient of τ^2

$$\tau^2: D_\beta v_2(\rho,\beta)=-a\left[v_0(\rho,\beta)\frac{\partial v_1(\rho,\beta)}{\partial\rho}+v_1(\rho,\beta)\frac{\partial v_0(\rho,\beta)}{\partial\rho}\right]$$
$$+c\frac{\partial^2 v_1(\mu,\beta)}{\partial\rho^2},$$

also coefficient of τ^3

$$\tau^3: D_\beta v_3(\rho,\beta)=\Bigg\{-a\Bigg[v_0(\rho,\beta)\frac{\partial v_2(\rho,\beta)}{\partial\rho}+v_1(\rho,\beta)\frac{\partial v_1(\rho,\beta)}{\partial\rho}+$$
$$v_2(\rho,\beta)\frac{\partial v_0(\rho,\beta)}{\partial\rho}\Bigg]+c\frac{\partial^2 v_2(\mu,\beta)}{\partial\rho^2}\Bigg\},$$

.

.

.

on integrate, we get

$$\tau^0: v_0(\rho,\beta)=u_0(\rho,\beta),$$

and

$$\tau^1: v_1(\rho,\beta)=I^\beta\left\{-\left(D_\beta u_0(\rho,\beta)+av_0(\rho,\beta)\frac{\partial v_0(\rho,\beta)}{\partial\rho}\right)+c\frac{\partial^2 v_0(\rho,\beta)}{\partial\rho^2}\right\},$$

also

$$\tau^2: v_2(\rho,\beta)=I_\beta\left\{-a\left(v_0(\rho,\beta)\frac{\partial v_1(\rho,\beta)}{\partial\rho}+v_1(\rho,\beta)\frac{\partial v_0(\rho,\beta)}{\partial\rho}\right)+c\frac{\partial^2 v_1(\rho,\beta)}{\partial\rho^2}\right\},$$

same the manner

$$\tau^3 : v_3(\rho,\beta) = I_\beta\left\{-a\left[v_0(\rho,\beta)\frac{\partial v_2(\rho,\beta)}{\partial\rho} + v_1(\rho,\beta)\frac{\partial v_1(\rho,\beta)}{\partial\rho} + \right.\right.$$

$$\left.\left. v_2(\rho,\beta)\frac{\partial v_0(\rho,\beta)}{\partial\rho}\right] + c\frac{\partial^2 v_2(\mu,\beta)}{\partial\rho^2}\right\},$$

.

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.

Once these values have been determined, the resulting value v can be acquired by substituting them into the power series (5), thereby yielding the sought-after solution.

Example (2):

The objective of this section is to apply HPM, led earlier, to crack a precise example of the generalized Burgers equation. Through the utilization of HPM, I aim to derive an approximate solution for this particular example of the generalized Burgers equation, as defined by (2) and (3).

We have

$$D_\beta v(\rho,\beta) = c\frac{\partial^2 v(\rho,\beta)}{\partial\rho^2}, \tag{14}$$

with the initial condition

$$v_0 = u_0 = \rho^2. \tag{15}$$

With the help of homotopy technique, we obtain

$$D_\beta v(\rho,\beta;\tau) = D_\beta u_0(\rho,\beta) - \tau\left(D_\beta u_0(\rho,\beta) - c\frac{\partial^2 v(\rho,\beta;\tau)}{\partial\rho^2}\right). \tag{16}$$

Let us consider

$$v(\rho,\beta;\tau) = \sum_{l=0}^{\infty}\tau^l v_l(\rho,\beta). \tag{17}$$

Now, using the same approach defined in above part, we obtain

$$\tau^0 : D_\beta v_0(\rho,\beta) = D_\beta u_0(\rho,\beta),$$

and coefficient of τ

$$\tau^1 : D_\beta v_1(\rho,\beta) = -D_\beta u_0(\rho,\beta) + c\frac{\partial^2 v_0(\rho,\beta)}{\partial\rho^2},$$

and coefficient of τ^2

$$\tau^2 : D_\beta v_2(\rho,\beta) = \frac{\partial^2 v_1(\mu,\beta)}{\partial \rho^2},$$

also coefficient of τ^3

$$\tau^3 : D_\beta v_3(\rho,\beta) = c\frac{\partial^2 v_2(\mu,\beta)}{\partial \rho^2},$$

.

.

.

By using the integration, we get

$$\tau^0 : v_0(\rho,\beta) = u_0(\rho,\beta),$$

and

$$\tau^1 : v_1(\rho,\beta) = I^\beta \left\{ c\frac{\partial^2 v_0(\rho,\beta)}{\partial \rho^2} \right\},$$

also

$$\tau^2 : v_2(\rho,\beta) = I_\beta \left\{ c\frac{\partial^2 v_1(\rho,\beta)}{\partial \rho^2} \right\},$$

same the manner

$$\tau^3 : v_3(\rho,\beta) = I_\beta \left\{ c\frac{\partial^2 v_2(\mu,\beta)}{\partial \mu^2} \right\},$$

.

.

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Once these values have been determined, the resulting value v can be acquired by substituting them into the power series (5), thereby yielding the sought-after solution.

Conclusions:

This document outlines an advanced approach aimed at investigating the generalized Burgers equation comprehensively by integrating its solution. The generalized Burgers equation holds significance in various fields, particularly in fluid dynamics and nonlinear partial differential equations. The study endeavors to delve deeper into this equation, seeking to establish a robust framework for understanding its behavior and solutions. The research commences by formulating theorems that are directly relevant to the generalized Burgers equation. These theorems serve as foundational principles upon which subsequent analyses are built. They provide a theoretical framework that guides the investigation and interpretation of the equation's properties and solutions. Subsequently, the study employs the homotopy perturbation method (HPM) as a primary analytical tool to analyze the generalized Burgers equation. The HPM is a powerful mathematical technique known for its effectiveness in solving nonlinear problems. By applying this method, the research aims to obtain approximate solutions to the generalized Burgers equation that accurately capture its behavior under various conditions. The analysis involves a systematic examination of the equation's output using the HPM. This includes iteratively solving the equation to obtain successive approximations and evaluating the convergence and accuracy of the obtained solutions. Through this process, the research aims to gain insights into the behavior of the generalized Burgers equation and the effectiveness of the HPM in providing analytical solutions. Furthermore, the document presents distinct outcomes derived from the application of the HPM to the generalized Burgers equation. These outcomes are meticulously analyzed and interpreted to highlight the efficacy of the method in approaching analytical solutions for this specific scenario. By showcasing these results, the research aims to demonstrate the practical utility and applicability of the HPM in studying nonlinear partial differential equations like the generalized Burgers equation. In summary, this research represents a comprehensive effort to investigate the generalized Burgers equation using an enhanced analytical approach. By integrating theoretical principles, mathematical techniques, and practical analyses, the study aims to contribute to a deeper understanding of the behavior and solutions of this fundamental equation in fluid dynamics and nonlinear physics.

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